Generalized coordinate

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1 Background

1.1 Where are we now?

In the last handout, we found an equivalent formulation of Newton's laws, *Energy*. This was a really handy shortcut for evaluating systems at given points in space, without having to integrate the equations of motion each time. But in many physics problems, we actually want to know how things are moving in time, to know the *dynamics*. Is there a way to do this as well using just ideas about energy, rather than forces? The answer is yes, and goes very deep in physics. You will see a smidge of this depth today.

1.2 Generalized coordinate

Let me show you a neat trick that is actually really deep. Let's say we have a system that we can describe with a single coordinate. It may be so that the system is made up of many objects (as in figure **??**), but they may be constrained in such a way that there is only one coordinate - "one degree of freedom", in fancy physics speak - necessary to describe it.

Let's call this coordinate ξ and call it a *generalized coordinate* to signal that it doesn't just refer to the coordinate of a single body, but that of an entire system. But you can consider it to be a coordinate like any other.

Then, we can write down the energy for the system:

$$E = \frac{M\dot{\xi}^2}{2} + V(\xi)$$

where $\dot{\xi}$ denotes the first derivative of ξ with respect to time, V the potential energy as a function of ξ and M the "mass". I put quotation marks around the mass, because as we will see this is not always equal to the actual total mass of the system, which is important.

Now, if we differentiate with respect to time once, using the chain rule, and then rearrange:

$$0 = M\dot{\xi}\ddot{\xi} + V'(\xi)\dot{\xi} \quad (1)$$

$$0 = M\ddot{\xi} + V'(\xi) \quad (2)$$

$$\ddot{\xi} = -\frac{V'(\xi)}{M} \quad (3)$$

$$(4)$$

So, we can find the accelerations (i.e. the dynamics) of the system by just considering how its energy will change. with respect to position. Magical! **This will be really useful in problems with complicated systems that have a lot of internal forces**, and will let us skip calculating all of those. Let's try this out on some problems!

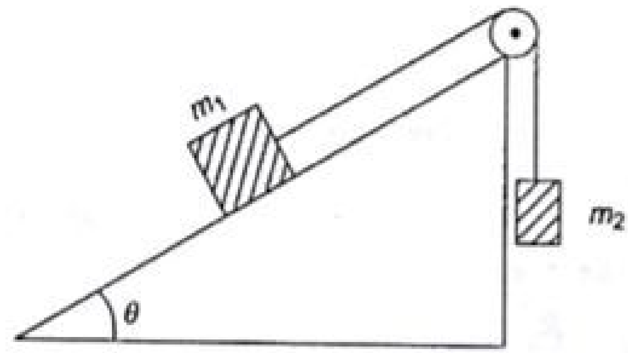

Figure 1: Blocks connected with rope over pulley

2 Questions

2.1 Check your understanding

Let's start by rederiving familiar results that we already know how to do with forces.

1. Consider the mathematical pendulum. That is, a pendulum consisting of a ball of mass m at the end of a massless rod of length l, which is in turn hinged at its opposite end and allowed to oscillate under gravity. Let the angle θ from the normal be the generalized coordinate describing the system.
   (a) Find the potential energy V of the pendulum as a function of θ.
   (b) Find the kinetic energy of the pendulum and express it as $\frac{M\dot{\theta}^2}{2}$. What is the expression for M?
   (c) Hence, find the angular acceleration of the mathematical pendulum, when it is an angle θ from the normal, using the results from a) and b).
2. Consider the physical pendulum. That is, a pendulum consisting of a uniform rod of mass $m$and length l, hinged at one end and allowed to oscillate under gravity. Let the angle θ from the normal be the generalized coordinate describing the system.
   (a) Find the potential energy V of the pendulum as a function of θ.
   (b) Find the kinetic energy of the pendulum and express it as $\frac{M\dot{\theta}^2}{2}$. What is the expression for M?
   (c) Hence, find the angular acceleration of the mathematical pendulum, when it is an angle θ from the normal, using the results from a) and b).
3. Find the acceleration of a block sitting on an incline with angle θ, using a generalized coordinate.

2.2 Trying out the waters

4. A massless chain hangs over a small smooth peg with equal lengths. Masses m and $3m$ are attached to the two ends, and the system is released from rest. Find the acceleration of the masses.
5. Two blocks of mass m_1 and m_2 are connected by a rope on a wedge of angle θ, as shown in figure 1. Find the acceleration of the blocks.
6. A wedge with acute angles α_1 and α_2 lies on a horizontal surface. A string has been drawn across a pulley situated at the top of the wedge, its ends are tied to blocks with masses m_1 and m_2. The wedge is fixed in place and cannot move on the table. What will be the acceleration of the wedge?
7. A small block of mass m lies on a wedge with angle α and mass M. The block is attached to a rope pulled over a pulley attached to the tip of the wedge and attached to a horizontal wall (see figure 2). Find the acceleration of the wedge. All surfaces are slippery (there is no friction). (Kalda)

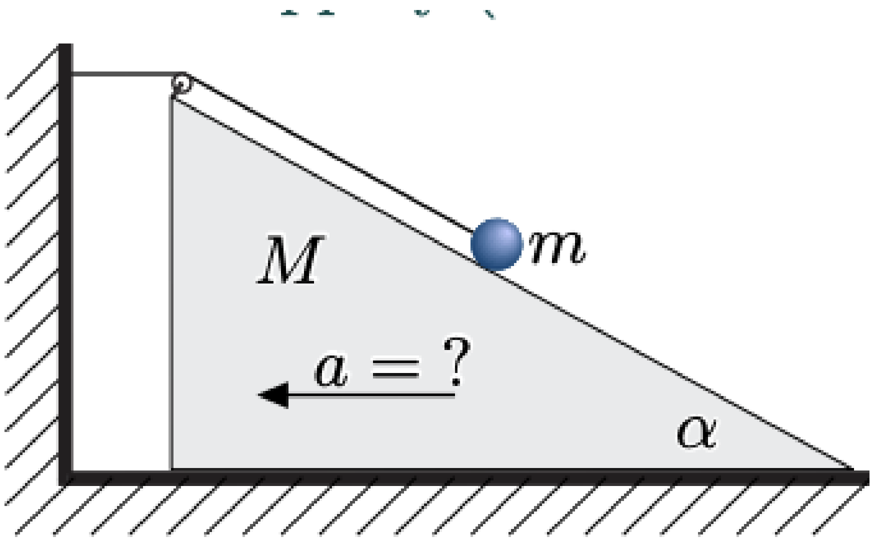

Figure 2: Ball with rope on incline

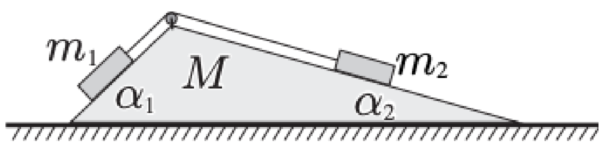

Figure 3: Masses on slanted wedge

2.3 Exploring the deep

Sometimes finding the constraints that link the coordinate of one object to another is not as straightforward as just using a constant separation (e.g. if they are tied together by a rope of fixed length). Instead, sometimes you can utilize the conservation of momentum to relate the velocity, and thus the positions, of different objects together.

8. Find the acceleration of a block m sliding down a larger block of mass M, with incline θ.

9. A wedge with mass M and acute angles α_1 and α_2 lies on a horizontal surface. A string has been drawn across a pulley situated at the top of the wedge, its ends are tied to blocks with masses m_1 and m_2.The wedge is now free to move. What will be the acceleration of the wedge? There is no friction anywhere.

10. Two slippery ($\mu = 0$) wedge-shaped inclined surfaces with equal tilt angles are positioned such that their sides are parallel, the inclines are facing each other and there is a little gap in between (see fig. 4). On top of the surfaces are positioned a cylinder and a wedge-shaped block, whereas they are resting one against the other and one of the block's sides is horizontal. The masses are, respectively, m and M. What accelerations will the cylinder and the block move with? Find the reaction force between them.

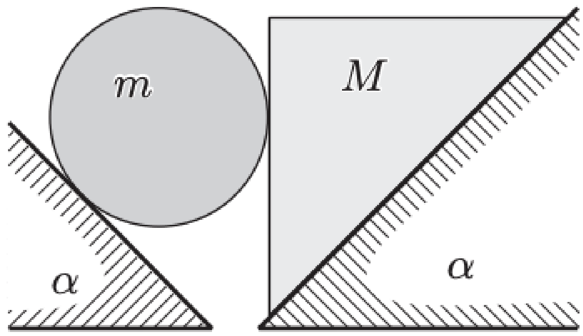

Figure 4: Sphere and block - how will they move?

Answers

1. a) $-mgl\cos\theta$ b) $M = ml^2$ c) $\alpha = -\frac{g\sin\theta}{l}$
2. a) $-\frac{mgl\cos\theta}{2}$ b) $M = \frac{ml^2}{3}$ c) $\alpha = -\frac{3g\sin\theta}{2}$
3. $g\sin\theta$
4. $a = \pm\frac{g}{2}$
5. $a = g\frac{m_2 - m_1\sin\theta}{m_1+m_2}$
6. $a = g\frac{m_2\sin\alpha_2 - m_1\sin\alpha_1}{m_1+m_2}$
7. $\frac{mg\sin\alpha}{M+2m(1-\cos\alpha)}$
8. $\frac{g\sin\theta}{\sin^2\theta+\frac{mM}{(m+M)^2}\cos^2\theta}$
9. $g\frac{(m_1\sin\alpha_1-m_2\sin\alpha_2)(m_1\cos\alpha_1+m_2\cos\alpha_2)}{(m_1+m_2+M)(m_1+m_2)-(m_1\cos\alpha_1+m_2\cos\alpha_2)^2}$
10. —